Introduction to Geometric Group Theory

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MAT-4112
(lecture note under construction)

Abstract

Geometric group theory studies groups from a geometric perspective. Given a finitely generated infinite group (such as $\mathbb{Z}^n$, free groups, surface groups, etc.), one constructs a metric space on which the group acts "nicely", and from the properties of this space one extracts properties of the group.

In this course, we will introduce some basic notions of geometric group theory and discuss a number of important examples of finitely presented groups. By the end of the course, students should be able to visualize these groups as geometric objects and recognize them through their geometric properties.

We will assume only basic knowledge of group theory (quotients, isomorphism theorems, ...) and of topology on metric spaces (connectedness, compactness, quotient spaces, ...). Some familiarity with algebraic topology would be helpful but is not required.

There will not be proofs for **Propositions** and **Corollaries** in this lecture note. They are also exercises! For each section, one **Theorem** will be attributed as homework.

We will roughly cover the textbook [Löh] *Geometric Group Theory: An Introduction* by Clara Löh, Chapters 1 to 7. Below are some useful references:

- **Undergraduate**
  - [Clay–Margalit] *Office Hours with a Geometric Group Theorists*
  - [Armstrong] *Groups and Symmetry*
- **Algebraic Topology**
  - [Massey] *A Basic Course in Algebraic Topology*
  - [Hatcher] *Algebraic Topology*
- **Graduate**
  - [Bridson–Haefliger] *Metric Spaces of Non-Positive Curvature*
  - [Druţu–Kapovich] *Geometric Group Theory*
  - [de la Harpe] *Topics in Geometric Group Theory*
  - [Lyndon–Schupp] *Combinatorial Group Theory*
  - [Serre] *Trees*
  - [Ol'shanskii] *Geometry of Defining Relations in Groups*
  - [ed. Ghys–Haefliger–Verjovsky] *Group Theory from a Geometrical Viewpoint*
- **French**
  - [Coornaert–Delzant–Papadopoulos] *Géométrie et Théorie des Groupes: Les Groupes Hyperboliques de Gromov*
  - [Ghys–de la Harpe] *Sur les Groupes Hyperboliques d'après Mikhael Gromov*
- **Gromov**
  - [Gromov 1987] *Hyperbolic Groups*, in *Essays in Group Theory*
  - [Gromov 1993] *Asymptotic Invariants of Infinite Groups*

Contents

1 Basics

1.1 Action!

1.1.1 Group actions on sets

A group action can be thought of as the "motion" of a space by a group.

Definition 1.1. *An **action** of a group G on a set X, denoted by $G \curvearrowright X$, is a function*

$$\alpha\colon G \times X \to X$$

where $\alpha(g,x)$ is written as $g \cdot x$ or gx, such that for all $g, h \in G$ and all $x \in X$,

- $1_G \cdot x = x$,
- $g \cdot (h \cdot x) = (gh) \cdot x$.

Equivalently, an action is a group homomorphism $\rho\colon G \to \mathrm{Sym}(X)$ where $\mathrm{Sym}(X)$ denotes the group of bijections of X, called the *symmetric group* of X; and $\rho(g)(x) = g \cdot x$.

Remark. What we have defined is a *left* action. We can also define a *right* action, denoted by $X \curvearrowleft G$, as a function $\alpha\colon X \times G \to X$ satisfying $x \cdot 1_G = x$ and $(x \cdot g) \cdot h = x \cdot (gh)$. A right action is then equivalent to an *anti*-homomorphism $\rho\colon G \to \mathrm{Sym}(X)$. That is, $\rho(gh) = \rho(h) \circ \rho(g)$.

Example. Some examples of group actions.

- $\mathrm{Sym}(X) \curvearrowright X$.
- $\mathrm{Sym}(X) \curvearrowright \mathcal{P}(X)$ where $\mathcal{P}(X)$ is the set of all subsets of X.
- $G \curvearrowright G$ by left multiplication, $G \curvearrowleft G$ by right multiplication.
- $\mathrm{Aut}(G) \curvearrowright G$.
- $\mathbb{Z} \curvearrowright \mathbb{R}$ by translation, $\mathbb{Z}^n \curvearrowright \mathbb{R}^n$ by translation.
- $\mathrm{Homeo}(\mathbb{R}) \curvearrowright \mathbb{R}$.
- $\mathbb{Z} \curvearrowright \mathbb{S}^1 = \{z \in \mathbb{C} \mid |z| = 1\}$ by rotation: $n \cdot z = e^{ni\theta} z$ where $\theta \in \mathbb{R}$.
- $\mathrm{GL}_n(\mathbb{R}) \curvearrowright \mathbb{R}^n$ by matrix multiplication.

1.1.2 Orbits, stabilizers, fixed points

Definition 1.2. *Given an action $G \curvearrowright X$.*

- *The **orbit** of $x \in X$ by G is the set*

$$\mathrm{Orb}_G(x) = G \cdot x := \{g \cdot x \mid g \in G\} \subset X.$$

- *The **stabilizer** of $x \in X$ by G is the set*

$$\mathrm{Stab}_G(x) := \{g \in G \mid gx = x\} \subset G.$$

Proposition 1.3. *Stabilizers are subgroups.*

Exercise. Show that $\mathrm{Stab}(gx) = g\,\mathrm{Stab}(x)g^{-1}$.
Exercise. Find orbits and stabilizers of $\mathrm{Sym}(X) \curvearrowright \mathcal{P}(X)$.

Definition 1.4. *Given $G \curvearrowright X$.*

- *The set of* ***fixed points*** *of $g \in G$ is the set*

$$\mathrm{Fix}(g) := \{x \in X \mid gx = x\} \subset X.$$

- *A* ***global fixed point*** *is an element $x \in X$ fixed by all $g \in G$. The set of global fixed points is then*

$$\bigcap_{g \in G} \mathrm{Fix}(g) = \{x \in X \mid \forall g \in G, gx = x\} \subset X.$$

Exercise. Find fixed points and global fixed points of $\mathrm{Sym}(X) \curvearrowright \mathcal{P}(X)$.

Theorem 1.5. *Let $G \curvearrowright X$ and $x \in X$. Denote by $G/\mathrm{Stab}(x)$ the set of left cosets of $\mathrm{Stab}(x)$ in G. Then the map*

$$\begin{aligned} G/\mathrm{Stab}(x) &\to \mathrm{Orb}(x) \\ g\,\mathrm{Stab}(x) &\mapsto gx \end{aligned}$$

is well-defined and bijective.

Proof. There are three things to be checked in exercises of this kind: well-definedness, injectivity, and surjectivity. In many cases one also needs to check that the map is a homomorphism of some structure, but this is not the case here.

- **Well-defined.** Suppose $g\,\mathrm{Stab}(x) = g'\,\mathrm{Stab}(x)$. Then $g^{-1}g' \in \mathrm{Stab}(x)$, hence $(g^{-1}g')x = x$. Multiplying by g on the left gives $g'x = gx$. Therefore, the image of a coset does not depend on the chosen representative.
- **Surjectivity.** Let $y \in \mathrm{Orb}(x)$. By definition of the orbit, there exists $g \in G$ such that $y = gx$. Thus y is the image of the coset $g\,\mathrm{Stab}(x)$.
- **Injectivity.** Suppose that $g\,\mathrm{Stab}(x)$ and $g'\,\mathrm{Stab}(x)$ have the same image, i.e. $gx = g'x$. Then $g^{-1}g'x = x$, so $g^{-1}g' \in \mathrm{Stab}(x)$, which implies $g\,\mathrm{Stab}(x) = g'\,\mathrm{Stab}(x)$.

□

Corollary 1.6 (Orbit-Stabilizer)**.** *If G is a finite group, then for any $x \in X$,*

$$|G| = |\mathrm{Stab}(x)||\mathrm{Orb}(x)|.$$

Exercise (Cauchy's Theorem). Let G be a finite group and let p be a prime dividing $|G|$. Let

$$X = \{(g_1, \ldots, g_p) \in G^p \mid g_1 g_2 \cdots g_p = e\}.$$

Define an action of $\mathbb{Z}/p\mathbb{Z}$ on X by

$$k \cdot (g_1, \ldots, g_p) = (g_{k+1}, \ldots, g_k).$$

- Show that every orbit of this action has size either 1 or p.
- Let $F \subset X$ be the set of global fixed points (i.e. of orbit size 1). Show that p divide $|F|$.
- Justify that $|F| = |\{g \in G \mid g^p = e\}|$. Conclude that there exists $g \in G$, $g \neq e$, such that $g^p = e$.

1.1.3 Free, transitive, faithful

Definition 1.7. *An action $G \curvearrowright X$ is said to be*

- ***free*** *if $gx \neq x$ for any $g \in G$ and $x \in X$;*
- ***transitive*** *if for any $x, y \in X$ there exists $g \in G$ such that $gx = y$;*
- ***faithful*** *if for any $g \in G$ there exists $x \in X$ such that $gx \neq x$.*

Exercise. Determine if $\operatorname{Sym}(X) \curvearrowright \mathcal{P}(X)$ is free, transitive, or faithful.

Proposition 1.8. *Every free action is a faithful action.*

Proposition 1.9. *If $G \curvearrowright X$ freely and transitively, then there is a natural bijection from G to X.*

Proposition 1.10. *An action $G \curvearrowright X$ is faithful if and only if the corresponding homomorphism $\rho\colon G \to \operatorname{Sym}(X)$ is a monomorphism. In any case, the action $G/\ker(\rho) \curvearrowright X$ defined by $g\ker(\rho) \cdot x := g \cdot x$ is well defined and faithful.*

Proposition 1.11. *Let $G \curvearrowright X$ be an action.*

- *The action is free if and only if for every $g \in G\backslash\{1_G\}$, $\operatorname{Fix}(g) = \varnothing$.*
- *The action is transitive if and only if there exists $x \in X$ such that $\operatorname{Orb}(x) = X$; if and only if for all $x \in X$, $\operatorname{Orb}(x) = X$.*
- *The action is faithful if and only if $\bigcap_{x\in X} \operatorname{Stab}(x) = \{1_G\}$.*

Theorem 1.12. *1. Every transitive action $G \curvearrowright X$ is "equivalent" to an action $G \curvearrowright G/H$ by left multiplication where H is a subgroup and G/H is the set of left cosets.*
2. $G \curvearrowright G/H$ and $G \curvearrowright G/K$ are "equivalent" if and only if H and K are conjugate in G.

Proof. Homework. □

Example (Dihedral group)**.**
The **dihedral group** D_n is the set of isometries of an n-gon, acting naturally on the n-gon. It consists of n rotations (including the identity) and n reflections. It's the subgroup of $\operatorname{Homeo}(\mathbb{S}^1)$ generated by $r\colon e^{i\theta} \mapsto e^{i\theta + i\frac{2\pi}{n}}$ and $s\colon e^{i\theta} \mapsto e^{-i\theta}$.

We can also define ∞-gon as the real line $\mathbb{R}$ where the points on $\mathbb{Z}$ are marked; and the **infinite dihedral group** D_∞ as the set of isometries of $\mathbb{R}$ that preserves $\mathbb{Z}$. It's the subgroup of $\operatorname{Homeo}(\mathbb{R})$ generated by $r\colon x \mapsto x + 1$ and $s\colon x \mapsto -x$.

Note that in either case, $srs = r^{-1}$.

Exercise. Let $n \in \{3, 4, \dots\} \cup \{\infty\}$. Consider the action of D_n on an n-gon, which consists of n vertices and n edges. Show that:

- D_n acts transitively on the set of vertices, but not freely.
- D_n acts freely on the set of pairs of vertices if n is odd and not freely if n is even, but never transitively.
- D_n acts transitively on the set of edge, but not free.
- D_n acts freely and transitively on the set of *oriented* edges.

1.2 Metric

1.2.1 Metric spaces

Definition 1.13. *A **metric space** (X, d) is a set X together with a distance function*

$$d \colon X \times X \to \mathbb{R}$$

such that for all $x, y, z \in X$,

Positive definite:	$d(x, y) \geq 0$ *and* $d(x, y) = 0$ *if and only if* $x = y$.
Symmetry:	$d(x, y) = d(y, x)$.
Triangle inequality:	$d(x, z) \leq d(x, y) + d(y, z)$.

Given $x \in X$ and $r > 0$, the open ball of radius r about x is the set

$$B(x, r) := \{y \in X \mid d(x, y) < r\},$$

and the closed ball

$$\overline{B}(x, r) := \{y \in X \mid d(x, y) \leq r\}.$$

Associated to the metric d one has the topology whose basis is the set of open balls $B(x, r)$. Note that in this topology, $\overline{B}(x, r)$ may be strictly larger than the closure of $B(x, r)$. The metric space is said to be **proper** if every closed ball $\overline{B}(x, r)$ is compact.

Given a metric space (X, d), a subset $Y \subset X$ is naturally a metric space $(Y, d_{|Y \times Y})$.

Example. The set $\mathbb{R}^n$ with the usual Euclidean metric

$$d(x, y) = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2}.$$

Example. Any set X together with the discrete metric

$$d(x, y) = \begin{cases} 0 & \text{if } x = y, \\ 1 & \text{if } x \neq y. \end{cases}$$

Compare $\overline{B(x, 1)}$ and $\overline{B}(x, 1)$.

Example. Let $\ell^2(\mathbb{R})$ denote the space of square–summable real sequences

$$\ell^2(\mathbb{R}) = \left\{ x = (x_n)_{n \geq 1} \;\middle|\; \sum_{n=1}^{\infty} x_n^2 < \infty \right\},$$

with the metric

$$d(x, y) = \sqrt{\sum_{n=1}^{\infty} (x_n - y_n)^2}.$$

Then $(\ell^2(\mathbb{R}), d)$ is *not* proper: the closed unit ball $\overline{B}(0, 1)$ is not compact, since the sequence $(e_n)_{n \geq 1}$ where $e_n = (0, \ldots, 0, 1, 0, \ldots)$ has no convergent subsequence.

1.2.2 Isometries and isometric actions

We now introduce the notion of *isometries*, that is, maps which preserve the metric structure of a space. This will allow us to define and study *isometric actions* on metric spaces.

Definition 1.14. *Let $f : X \to X'$ be a function from one metric space (X, d) to another (X', d').*

- *We say that f is an* ***isometric embedding*** *if*

$$d'(f(x), f(y)) = d(x, y) \quad \textit{for all } x, y \in X.$$

- *In addition, if there exists another isometric embedding $g : X' \to X$ such that*

$$g \circ f = Id_X \text{ and } f \circ g = Id_{X'},$$

  then we say that f is an ***isometry****.*

- *The two metric spaces (X, d) and (X', d') are said to be isometric.*

- *The set of isometries if a metric space (X, d) is denoted by $\mathrm{Isom}(X)$.*

Proposition 1.15.

- *An isometric embedding is injective and continuous.*
- *A surjective isometric embedding is an isometry.*
- *$\mathrm{Isom}(X)$ is a subgroup of $\mathrm{Homeo}(X)$, the set of homeomorphisms of X.*

Example. Let $m \leq n$ be integers. Then the canonical inclusion $\mathbb{R}^m \hookrightarrow \mathbb{R}^n$ is an isometric embedding.

Example. The map $f : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $f(x, y) = (y, x)$ is an isometry.

Definition 1.16. *An isometric action of a group G on a metric space (X, d) is a group homomorphism $\rho : G \to \mathrm{Isom}(X)$..*

We say that G acts on X *by isometries* or *isometrically*.

Exercise. Let $G \curvearrowright (X, d)$ by isometries. Show that $gB(x, r) = B(gx, r)$ for any $g \in G$, $x \in X$, and $r > 0$.

Definition 1.17. *Let $G \curvearrowright (X, d)$ by isometry. The action is said to be*

- ***proper*** *if for any $x \in X$ and any $r > 0$, the set $\{g \in G \mid gB(x, r) \cap B(x, r) \neq \varnothing\} \subset G$ is finite.*
- ***cobounded*** *if there exists $x \in X$ and $r > 0$ such that $G \cdot B(x, r) = X$.*
- ***cocompact*** *if there exists a compact set K such that $G \cdot K = X$.*

Example. $\mathbb{Z}^n \curvearrowright \mathbb{R}^n$ by translations is proper and cocompact.

Exercise. Find a free action that is not proper, and a proper action that is not free.

Exercise. Show that a transitive action is cocompact, and a cocompact action is cobounded.

Exercise. Show that if the metric space is proper, then a cobounded action is cocompact.

1.2.3 Geodesic metric spaces

Definition 1.18 (Geodesic metric space)**.** *Let (X, d) be a metric space. The set of real numbers $\mathbb{R}$ is endowed with the usual metric $d_{\mathbb{R}}(a, b) = |a - b|$.*

- *A **geodesic** of X is a map γ from an interval $I \subset \mathbb{R}$ to X which is an isometric embedding. That is, $d(\gamma(a), \gamma(b)) = |a - b|$ for every $a, b \in I$.*

  If $I = [a, b]$ is finite, we say that the points $\gamma(a), \gamma(b) \in X$ are joined by the geodesic γ.
- *Let $L > 0$. An L-**local geodesic** of X is a map γ from an interval $I \subset \mathbb{R}$ to X such that for every sub-interval $J \subset I$ of length at most L, the restriction $\gamma_{|J}$ is a geodesic.*
- *The space X is called a **geodesic metric space** if every pair of points can be joined by a geodesic.*

Example. A great circle on a sphere is a local geodesic but not a geodesic.

Example. $\mathbb{R}^n$ with the Euclidean metric is a geodesic metric space, $\mathbb{R}^n \backslash \{0\}$ is not.

1.3 Graphs and Cayley graphs

1.3.1 Graphs

For the moment we consider graphs as 1-dimensional simplicial complexes, not oriented, without loops or multiple edges. For graphs as combinatorial 1-dimensional cell complexes, see [Lyondon Schupp] or [Hatcher]. For graphs in the sense of Serre, see [Serre].

Definition 1.19. *A graph is a pair $\Gamma = (V, E)$ of disjoint sets where E is a set of subsets of V that contain exactly two elements. That is,*

$$E \subset \{e \mid e \subset V, |e| = 2\}.$$

A graph is **finite** if V and E are both finite. The **degree** (or **valence**) of a vertex $v \in V$, denoted by $\deg(v)$, is the number of its appearance in the edges. A graph is **locally finite** if every vertex has finite degree.

Exercise. For a finite graph,

$$\sum_{v \in V} \deg(v) = 2|E|.$$

Some vocabularies:

- A **path** of length n is a sequence of vertices $v_0, v_1, \ldots, v_n$ such that $\{v_i, v_{i+1}\} \in E$.
- A **cycle** of length n is a path $v_0, v_1, \ldots, v_{n-1}$ such that $\{v_{n-1}, v_0\} \in E$.
- A **simple** path (resp. cycle) is a path (resp. cycle) whose vertices are different.
- A graph is **connected** if every pair of vertices can be connected by a path.
- A **tree** is a connected graph with no cycles.
- A graph is d-**regular** for some integer $d \geq 1$ if every vertex has degree d.

Example. Some examples of graphs.

- A cycle of length n.